The Art and Science of Hyperbolic Tessellations

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The visual impact of hyperbolic tessellations has captured artists' imaginations ever since M.C. Escher generated his Circle Limit series in the 1950s. The scaling properties generated by hyperbolic geometry are different to the fractal scaling properties found in nature's scenery. Consequently, prevalent interpretations of Escher's art emphasize the lack of connection with nature's patterns. However, a recent collaboration between the two authors proposed that Escher's motivation for using hyperbolic geometry was as a method to deliberately distort nature's rules. Inspired by this hypothesis, this year's cover artist, Ben Van Dusen, embeds natural fractals such as trees, clouds and lightning into a hyperbolic scaling grid. The resulting interplay of visual structure at multiple size scales suggests that hybridizations of fractal and hyperbolic geometries provide a rich compositional tool for artists.

Key words: Fractals, hyperbolic tessellations, M.C Escher.

It has become popular to view the spectrum of disciplines as a circle, with mathematics and art lying so far apart that they become neighbors. Two artists are celebrated as proof of this theory - Leonardo da Vinci (1452-1519) and Maurits Escher (1898-1972). Da Vinci combined mathematics and art to search for the possible, resulting in functional designs such as his famous flying machines. In contrast, Escher searched for the impossible by creating images that distorted nature's rules. Escher's prints of tessellations were inspired by the Islamic tiles he saw during a trip to the Alhambra in Spain [Escher, 1989]. However, he took the bold step of incorporating patterns that repeat at many size scales. To achieve the desired visual balance, he insisted that the shrinking patterns converge towards a circular boundary. In Escher's words, the repeating patterns emerged from the circular boundary "like rockets", flowing along curved trajectories until they "lose themselves" once again at the boundary [Ernst, 1995]. Making his patterns fit together required considerable thought and a helping hand from mathematics. After several flawed attempts, Escher finally found the solution in an article written several years earlier by the British geometer H.S.M. Coxeter [Coxeter, 1979, 2003]. In Figure 1 (a, b) we show two examples of 'Circle Limit' tessellations. Van Dusen generated these images using Escher's rules of increasing the tessellation sizes as a function of the distance from the outer circular boundary.

The flowing patterns of Circle Limit tessellations have captured the imaginations of both artists and mathematicians for over half a century. Yet along the way, their connection with nature has fallen by the way side [Taylor, 2009]. They are often presented as an elegant solution to a purely academic exercise of mathematics - a clever visual game. In reality, Escher's interest lay in the fundamental properties of patterns that appear in the real world. He declared: "We are not playing a game of imaginings – we are conscious of living in a material three dimensional reality." [Ersnt,1995]. Escher's artistic interest in the physical world is emphasized by the sketches of trees completed in the same era as his Circle Limit tessellations [Escher, 1989]. His sketches capture how branch patterns repeat at different size scales and how they become distorted when reflected in the rippled surface of a pond. Given Escher's quest for distortion, it comes as no surprise to find that his tessellations deviate from the fractal scaling properties of nature's patterns, which will be discussed in more detail later.

Van Dusen's art is designed to celebrate the inherent distortion in the tessellations created by Escher. In this article, we will show that Van Dusen's Circle Limit tessellations shrink at a different rate to patterns found in nature. Specifically, Circle Limit tessellations employ the hyperbolic geometry described in Coxeter's article [Coxeter, 1979, 2003], rather than the fractal geometry that describes natural patterns [Mandelbrot, 1982]. To emphasize the importance of distortion, Van Dusen's tessellations play a subtle visual trick on the observer. Escher employed simple symbols such as fish, bats and reptiles to form the basic tiling patterns, which he then shrank according to the rules of hyperbolic scaling. In contrast, Van Dusen uses nature's fractal objects – such as the mud cracks and clouds shown in Figure 1 – as the basic tiling patterns which he then shrinks using hyperbolic scaling. Before seeing the results, one might expect this clash of scaling phenomena to generate a visual battlefield. After all, why should nature's fractals sit nicely within hyperbolic scaling rules? The fact that they do so highlights the success of Escher's artistic mission – to show that art based on distortions can be just as striking as those that follow nature's scaling rules with fidelity. In this essay, we will first describe how Circle Limit tessellations are generated and then perform a scaling analysis to quantify the visual impact of Van Dusen's intriguing integration of scaling forms.

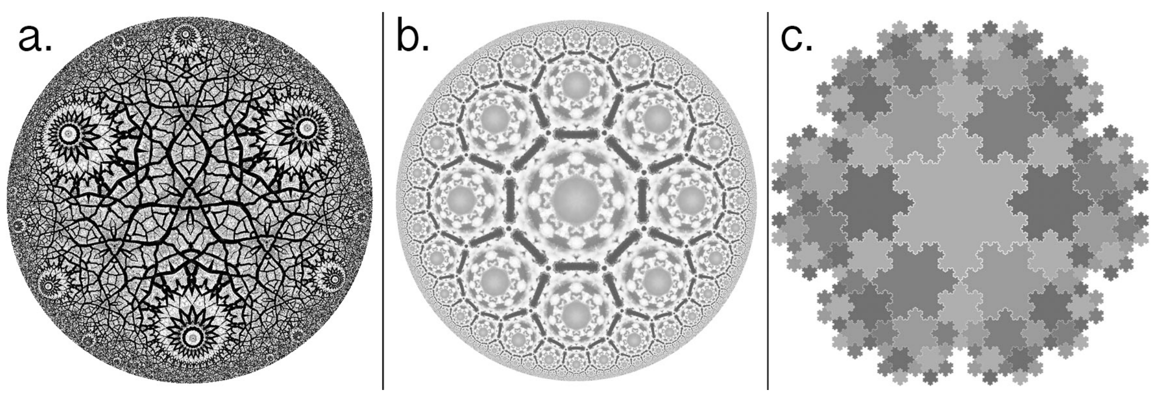

Figure 1: a) Van Dusen's Parched Earth, showing mudcracks embedded within a hyperbolic grid b) Van Dusen's Summer Sky, showing clouds embedded in a hyperbolic grid, c) A version of a Koch Snowflake generated for direct comparison with Circle Limit tessellations.

Tessellations are patterns that fill the surface of a plane without any overlaps or gaps. Most tessellation designs in art feature component tiles which are identical in size. In contrast, in *Circle Limit* artworks the size of the tessellations diminish towards the infinitesimally small as the circular edge is approached. To create this visual effect, it is necessary to generate tile shapes that are not typically tiled together. When the basic shapes used in *Circle Limit* artworks are tiled together they don't create a flat surface. Instead of fitting together evenly in two dimensions, the pieces instead form the hyperbolic geometry shown in Figure 2. A hyperbolic geometry looks much like a saddle: on one axis, the surface rises upward form the origin (symbolized by the dot) and on the other axis the surface drop downward. Significantly, when this hyperbolic surface is viewed directly from above, it appears to spread out indefinitely and any tiles 'drawn' on the surface become more distorted the further they get from the origin. Thus, although all of the tiles have equal size on the surface, they appear to shrink towards the edge.

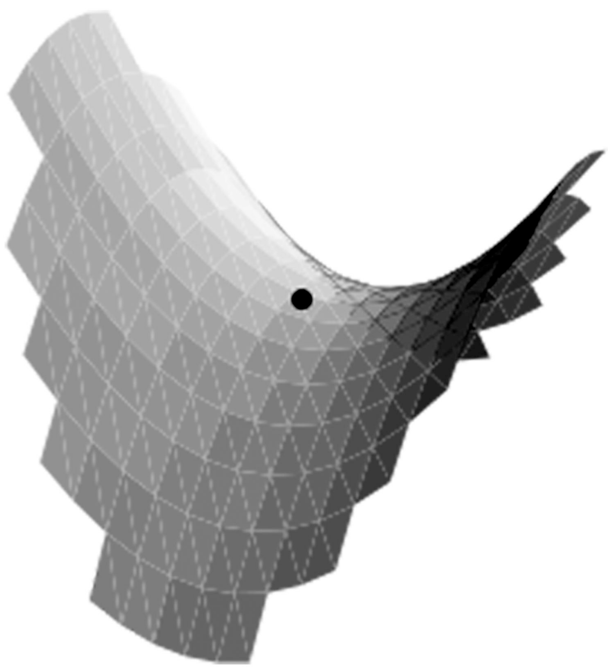

Figure 2: Hyperbolic geometries form the shape of a saddle with the surface rising along one axis and dropping along the other axis.

To make the entire surface viewable, the surface has to be translated onto a Poincaré disk [Anderson, 2005]. A Poincaré disk is a circle that represents an infinite region of space. As the circular edge is approached, the images diminish at such a rate that they appear to be infinitely small and be infinitely close to the circle's edge without ever touching it. By using this Poincaré disk model, it is possible to give the impression of an infinite array of tile images within a limited space and, unlike other disk models, the shape of the tiles stays recognizable as they approach the circular boundary. As shown in Figure 3(a), a tessellation of octagons forms the basis of Escher's and some of Van

Dusen's Circle Limit patterns. Figure 3(b) shows the visual effect when Van Dusen embeds fractal patterns into this hyperbolic tiling pattern: each octagon features the fractal branches of a tree.

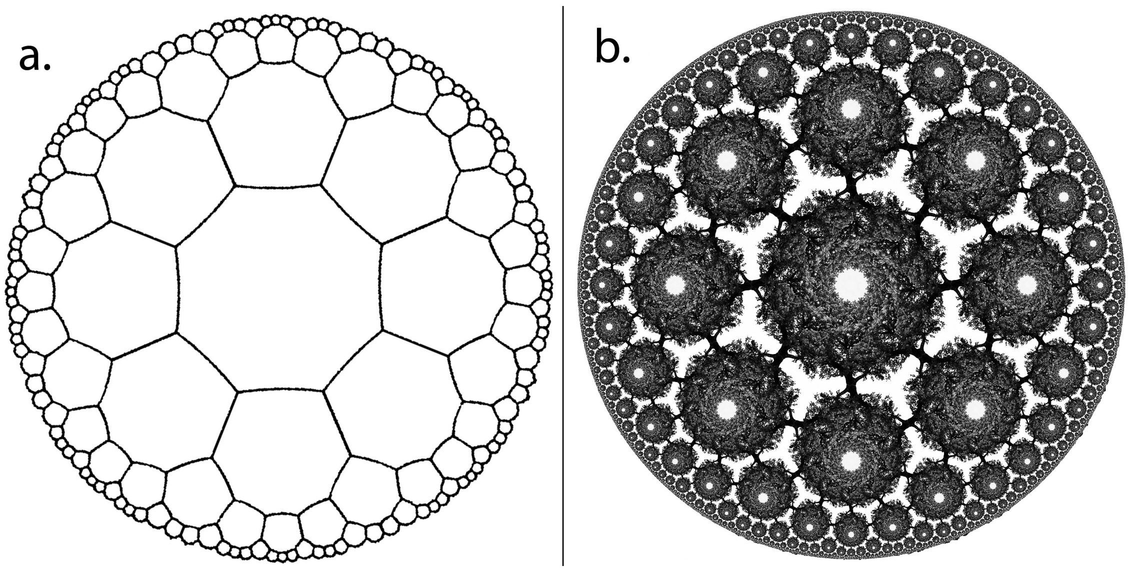

Figure 3*:* a) the hyperbolic scaling grid used in octagonal Circle Limit artworks, b) the hyperbolic scaling grid with a fractal tree pattern embedded.

To highlight the visual differences between a pure fractal and Van Dusen's hyperbolic-fractal hybridizations, we created a purely fractal pattern featuring interlocking tessellations, as shown in Figure 1(c). Based on the Koch curve, the pattern consists of triangles that repeat at increasingly fine scales, thereby building up the edge of a snowflake [Koch, 1904]. The visual difference between these two scaling rates is highlighted in Figure 4, where the edge patterns for the Van Dusen and Koch tessellation designs have been isolated. This isolation of the edge patterns allows their scaling behavior to be analyzed using the box counting technique. Adopting this technique, the image of white edges is covered with a computer-generated mesh of identical squares. By analyzing which of the squares are occupied (i.e., contains a part of the white edge pattern) and which are empty, the statistical qualities of the edge pattern can be calculated. Specifically, if N, the number of occupied squares, is counted as a function of L, the square size, then for fractal behavior $N(L)$ scales according to the power law relationship $N(L) \sim L^{-D}$ [Mandelbrot, 1982, Taylor, 2011]. The exponent D is called the fractal dimension and its value can be extracted from the gradient of the scaling plot of log(N) against log($1/L$).

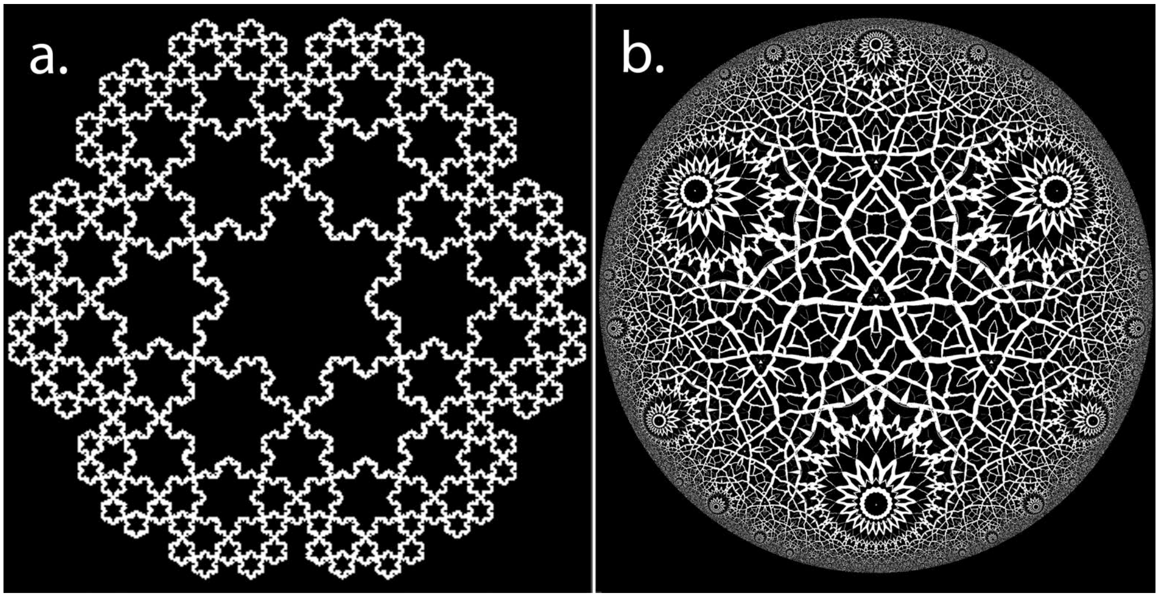

Figure 4: The edge patterns of a) the Koch Snowflake, and b) Van Dusen's Parched Earth

The data for the Koch Snowflake, shown in Figure 5, displays the straight power-law line expected for a fractal pattern. In contrast, the data for Van Dusen's Parched Earth fails to condense onto a straight line. To interpret the visual significance of this curved line, we need to consider the importance of the fractal's power law line, as quantified by the gradient *D*. Traditional measures of visual patterns quantify complexity in terms of the ratio of fine structure to course structure. *D* goes further by quantifying the relative contributions of the fractal structure at all the intermediate magnifications between the course and fine scales. For example, a high *D* value is a signature of larger *N* values at smaller *L* values and reflects a high visual complexity. How, then, do we interpret the visual impact quantified by the curved data line for Van Dusen's Parched Earth? Because of its curvature, the data line can't be quantified by a *D* value. Nevertheless, the steepness of the line holds the same visual consequences as for a fractal pattern. The reduction in gradient observed for the Circle Limit pattern at fine scales (i.e. large log $(1 / L)$) values generates a lower *N* value at those fine scales than for the fractal pattern. Thus, the fine structure in the Circle Limit pattern covers less space than those of the fractal patterns. In other words, the patterns in the Circle Limit pattern diminish in size (i.e. space-coverage) at a faster rate than those of the Koch Snowflake. An inspection of Figures 1 and 4 confirms that this is indeed the case.

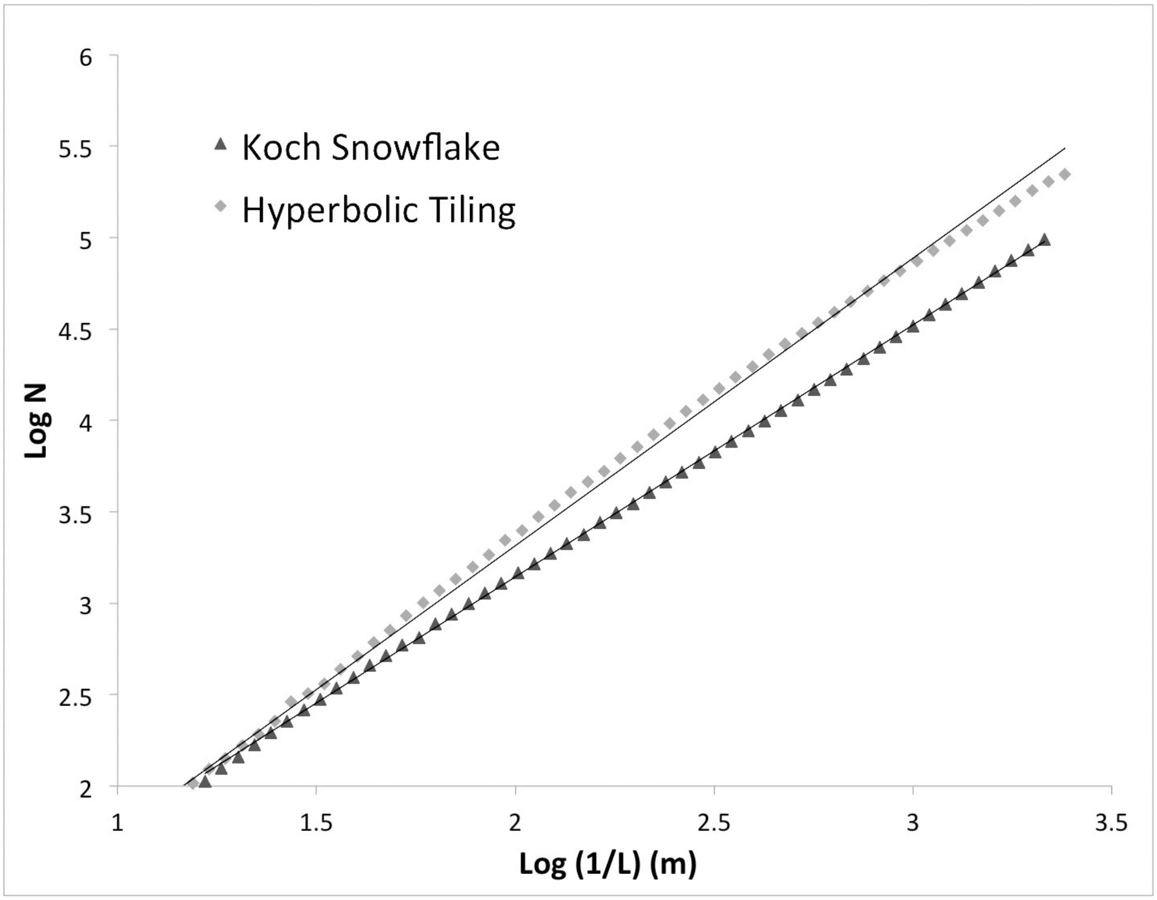

Figure 5: Scaling plots obtained from the box counting analysis of the Koch Snowflake (dark gray data) and Van Dusen's Parched Earth (light gray data). The analyzed images had dimensions 42 cm by 42 cm and resolution of 63 pixels per cm.

This is the essential visual difference between the fractal patterns of nature and Van Dusen's tessellations – nature's fractal patterns decrease at a constant rate set by the power law behavior, while Van Dusen's tessellations diminish at an accelerated rate set by the hyperbolic surface in which his fractal patterns are embedded. This is confirmed by the fact that 'pure' hyperbolic tessellations (i.e. ones which feature simple, non-fractal shapes as their basic tilting component) reveal a similar scaling curve to Van Dusen's hybrid tessellations [Van Dusen, 2012]. Significantly, perception experiments reveal that fractals with mid-range D values between $D = 1.3 – 1.5$ are aesthetically pleasing to the observer [Aks, 1996, Spehar, 2003, Taylor, 2011]. It would be interesting to extend studies of human perception of multi-scale stimuli to include the hyperbolic patterns of Escher and the hybrid patterns of Van Dusen. Based on Escher's artistic goal of distorting nature's patterns, we might expect to find interesting aesthetic preferences to be triggered by these curious creations.

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