An Optical Demonstration of Fractal Geometry

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Abstract

We have built a Sinai cube to illustrate and investigate the scaling properties that result by iterating chaotic trajectories into a well ordered system. We allow red, green and blue light to reflect off a mirrored sphere, which is contained in an otherwise, closed mirrored cube. The resulting images are modeled by ray tracing procedures and both sets of images undergo fractal analysis. We offer this as a novel demonstration of fractal geometry, utilizing the aesthetic appeal of these images to motivate an intuitive understanding of the resulting scaling plots and associated fractal dimensions.

Nature abounds with fractal scaling patterns. Few would contest Mandelbrot's classic argument "Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line"[1]. Today researchers find fractal geometry a useful description of a wide variety of phenomena, as anyone can verify on the internet within a few minutes using their favorite search engine. One is likely to find a variety of subjects using fractal geometry to describe anything from pulmonary vessels to river systems to a wide range of artwork.

Often when one thinks of fractals however, the images that come to mind are of those of exact fractals, usually generated by repetitive application of mathematical formulae. With the use of computer simulations truly beautiful images can be generated. Two canonical examples of these iteratively generated fractals are the Koch snowflake and the Sierpinski gasket. In the case of the Koch snowflake a line segment is divided into thirds the middle third of the line is replaced by two equal length segments angled to form the top of an equilateral triangle. In the next iteration of this process each of the 4 new segments is replaced by a 1/3 replica of the parent segment according to the same procedure.[2] At each iteration step we are increasing the overall length of the curve by a factor of 4/3. This curve can be characterized by the fractal dimension *(D)* as follows. If one takes a line segment and chops it into *N* identical length segments *L*, the line is said to be reduced by a factor $L = 1/N$. If one similarly segments a square the scaling factor can be described by $L = 1/\sqrt{N}$ for a cube the analogous process yields $L = 1/\sqrt[3]{N}$. We can generalize this expression to a *D* dimensional object by the expression $L = 1/\sqrt[D]{N}$. Thus each of the *N* copies of the original image each have a length *L* and there are $N = 1/L^D = L^{-D}$ segments of the original object. This gives the fractal dimension of self similar objects as $D = \log(N)/\log(1/L)$. For the Koch snowflake we have 4 copies scaled down by a factor of 1/3 giving $D = \log(4)/\log(3) \approx 1.26$. A similar process can be used to generate the Siepinski gasket. Starting with an equilateral triangle we cut out an inverted triangle that has been scaled down by a factor of *L*= ½ leaving *N*=3 smaller copies of the original triangle thus for the Sierpinski gasket $D = \log(3)/\log(2) \approx 1.58$. Fractals generated by such recursive processes are members of a class of fractals known as Iterated Function Systems (ITS). Figure 1 shows the original image followed by the first two iterations of each of these processes.

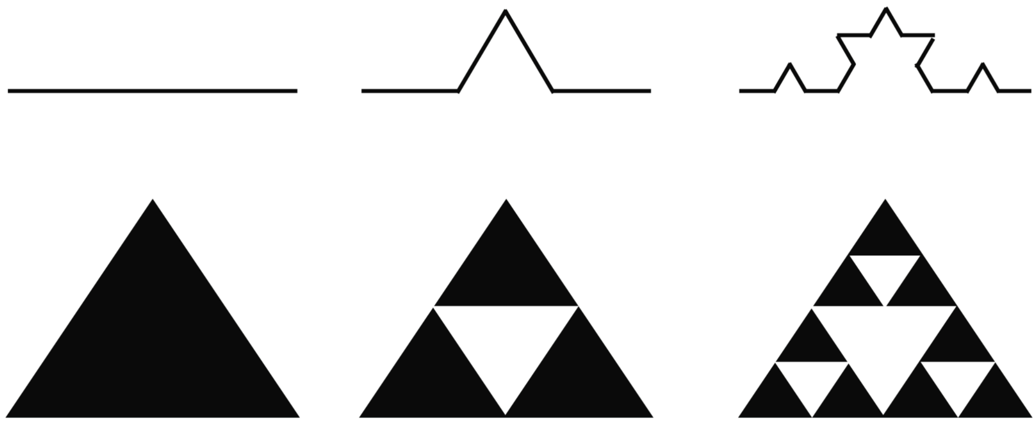

Figure 1: *The starting point and first two iterations of the generation processes of the (top) Koch snowflake and (bottom) Sierpinski gasket.*

An intuitive feel for what D actually characterizes can be rather difficult to grasp. One possible interpretation that works well for coastlines or curves such as the Koch snowflake is that D can be viewed as a measure of the crinkliness of the curve or its lack of smoothness. This interpretation serves well for as a description of D provided that it is the edges of a pattern that result in the fractal scaling. But such a description is lacking for an intuitive feel to filled patterns such as the Sierpinski gasket.

While analytic determination of D is fitting or at least possible for fractals generated via IFS, difficulty arises when one encounters a fractal produced by natural processes. To estimate D for these processes typically one employs the box counting procedure. In this procedure one superimposes a grid of boxes of length L on a two dimensional image of the object in question and counts the number of boxes that contain a piece of the object N. The box size is then decreased and the counting of filled boxes is repeated until a suitably large range of L has been spanned. One can then produce a scaling plot in which $\log(N)$ is plotted as a function of $\log(1/L)$ and D is determined by the slope of the curve. In this scenario an object is said to be fractal if $1 < D < 2$, provided that the expression $N \propto L^{-D}$ holds for at least 1 order of magnitude.

Figure 2 shows an example of the scaling plots of the Sierpinski gasket and the Koch curve. When plotted in this manner it is convenient to interpret the $\log(1/L)$ as describing the box size as ranging from coarse scaling in which just a few boxes span the image to fine scaling where the number of boxes necessary to span the image becomes quite large.

In this way one can picture the value of D as a measure of the fine scale structure of the pattern. We see in Figure 2 that the Koch curve (bottom inset) does not seem to have as much complexity at fine scales as the Sierpinski gasket (top inset). The fact that the Sierpinski gasket fills more fine scale boxes is readily evident in that N is larger for the Sierpinski gasket than the Koch curve at the fine scale end of the plot while both curves are anchored at the same coarse scale values. In some sense this implies that one can view D to be a measure of the ability of the object to occupy space at fine scales.

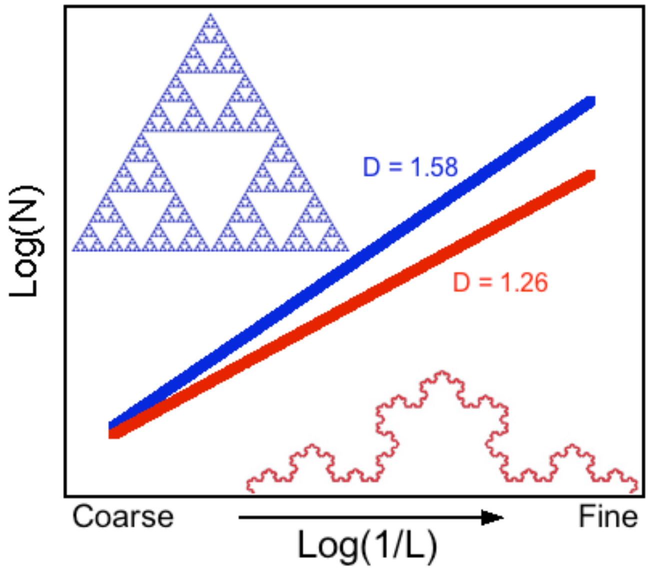

Figure 2: *Scaling plots generated by the analytic value of the fractal dimension of the Sierpinsk gasket (top scaling plot with D =1.58 and inset) and the Koch snowflake (bottom scaling plot with D= 1.26 and inset).*

For the past few years work in our lab has focused on fractal scaling properties of conductance fluctuations in semiconductor billiards. In 1996 Ketzmeric published a paper illustrating that the mixed (chaotic and regular) phase space of chaotic systems would generate fractal trajectories.[3] He went on to predict that the conductance fluctuations of semiconductor nanostructures would exhibit this fractal scaling. In 1997 Taylor *et al.* reported the first experimental observation of self similarity in the so called Sinai billiard.[4] Figure 3 illustrates the transition from regular to chaotic dynamics that emerge as a result of altering device geometry.

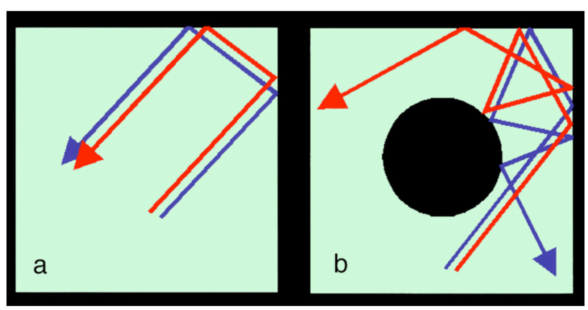

Figure 3: *a) A square billiard exhibits regular dynamic, trajectories with similar initial conditions do not diverge. B) Introducing a circular scatterer (Sinai diffuser) creates trajectories that are exponentially sensitive to their initial conditions, thus introducing chaos to the system.*

A few years ago Marlow *et al.* presented a unified view of these conductance fluctuations based on a comprehensive comparison of several device regimes, showing that these fractal scaling patterns in the conductance fluctuations were not dependent on the device geometry or scattering regime properties that Ketzmeric's proposal did not explain.[5] We offered a unified model, proposing that the electrostatic influence of remote ionized regions acted as Sinai diffusers in the devices. In essence, the fractal patterns resulted from the device walls allowing repeated reflection off of the diffusers, iterated chaos akin to the IFS mentioned above.

Coupling this idea with our lab's dedication to physics education, we set out to develop a simple demonstration of this phenomenon suitable for use in the classroom. Intrigued by the apparatus described by Sweet *et al.,*[6] we first built a light scattering system consisting of four reflective spheres stacked in a pyramid formation. Replicating Sweet's experiment we illuminate the top three openings created by the stacked configuration with colored light. The fourth (bottom) opening is used to capture an image of the pattern created by the iterated reflections. This apparatus displays the properties of Sinai diffusers in the absence of wall boundaries. To investigate the influence of the wall on this sort of system we built a Sinai cube. This also had the advantage of being a classical analog of Sinai diffuser model of our electron billiards. Using front surface mirrors we constructed a set of cubes with openings at the upper corners. We suspend a reflective sphere from the top mirror creating the Sinai cube. Three of the top corners are illuminated with colored light, and in the fourth corner we place a camera to capture the image created by the lights repeated reflection off of the Sinai diffuser. In Figure 4 we show these apparatuses.

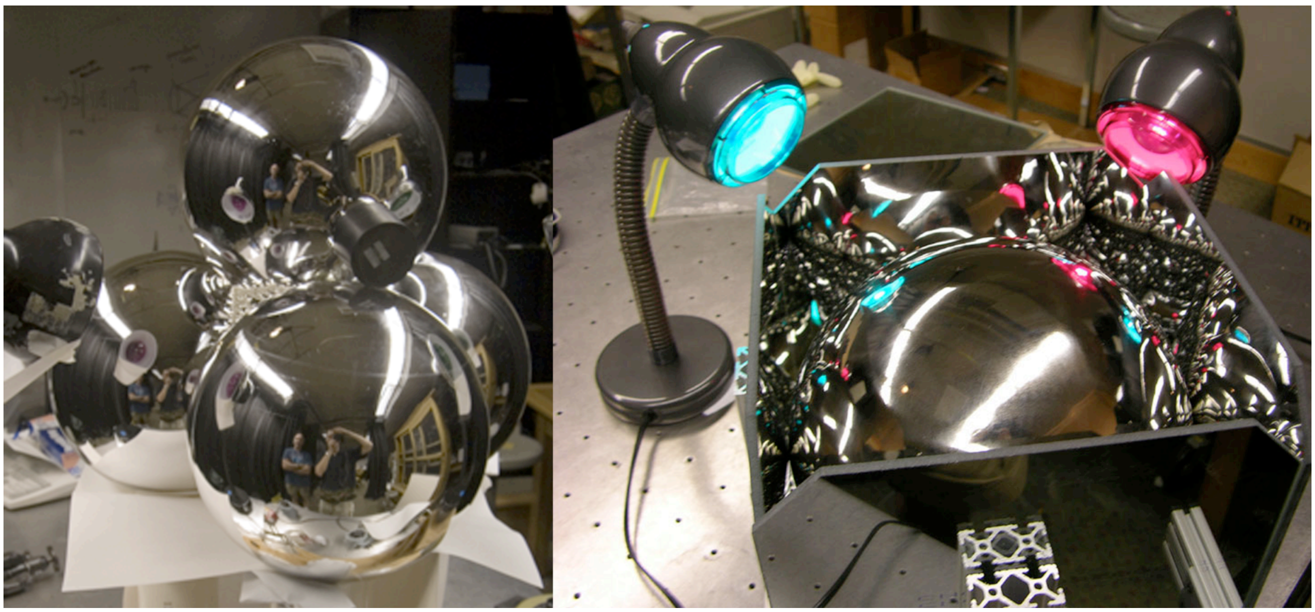

Figure 4: *The two experimental apparatuses are pictured above. On the left is a recreation of the laboratory model presented by Sweet et al. On the right is our Sinai cube shown here without the top mirror in place and with a Sinai diffuser whose diameter is approximately equal to the wall width.*

We use ray tracing techniques[7] to model both the stacked sphere and Sinai cube images with good results. Figure 5 shows the image obtained from the bottom opening of the stacked sphere configuration (top) as well as the ray tracing model (bottom). In Figure 6 we show the experimental (top) and modeled (bottom) Sinai cube in which the diffuser diameter is roughly 1/3 the wall width. A box counting analysis was performed on both sets of images with good agreement with the values reported by Sweet *et al.* for the stacked spheres (D= 1.6) and good agreement between the experimental and modeled images of the Sinai cube.

Taking a step back and regarding D in terms of the fine scale space filling interpretation, we are observe that the resulting D values are in line with this description as depicted in Figure 7 where we show the scaling plot of the 33% filled Sinai cube model and image (upper solid and dashed lines respectively). The lower solid line depictis the agreed upon value (D=1.6) of the stacked sphere configuration (lowest solid line).

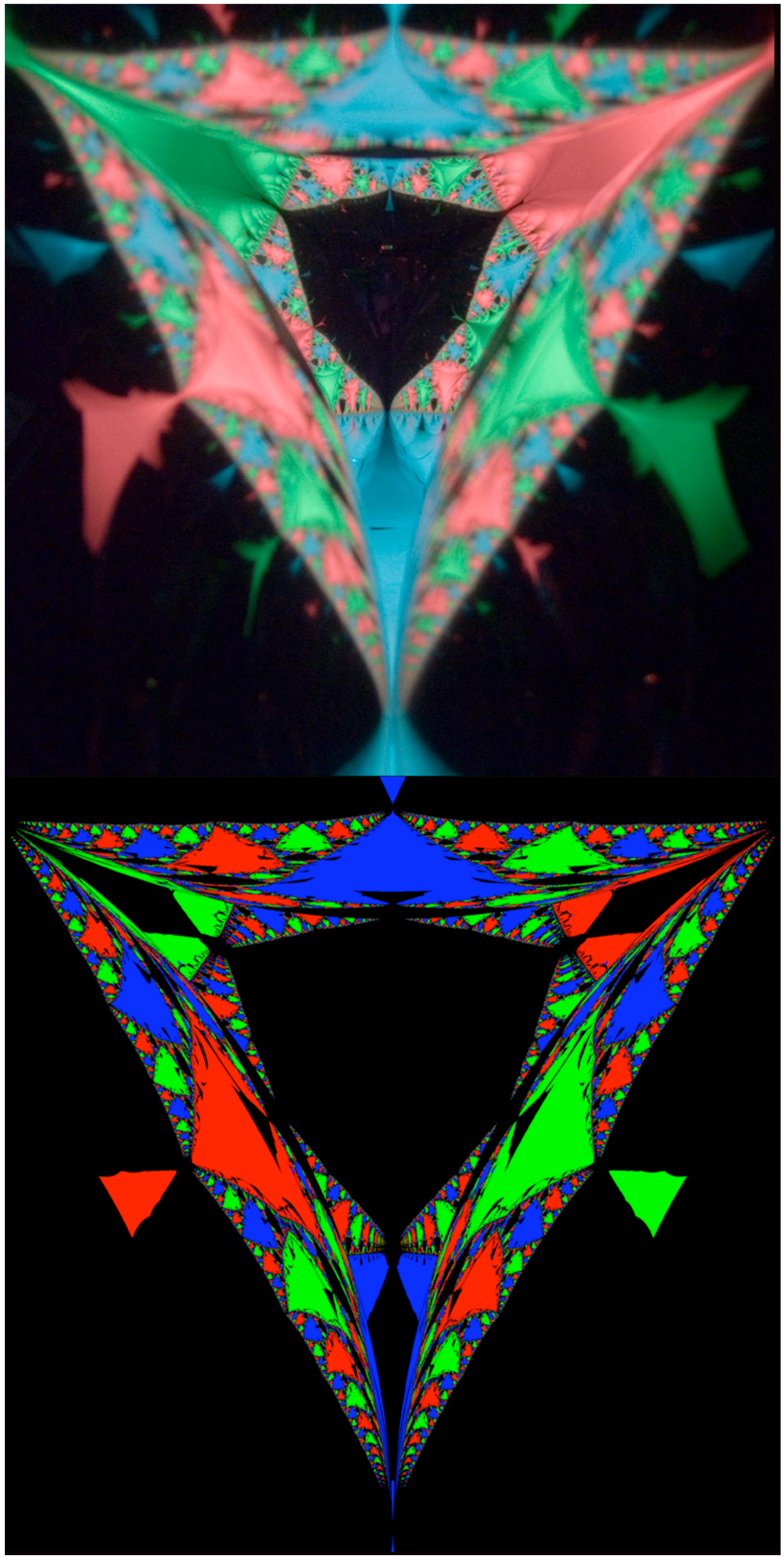

Figure 5: *(top) The image obtained from the bottom opening of the stacked sphere configuration and (bottom) the ray tracing model of that image.*

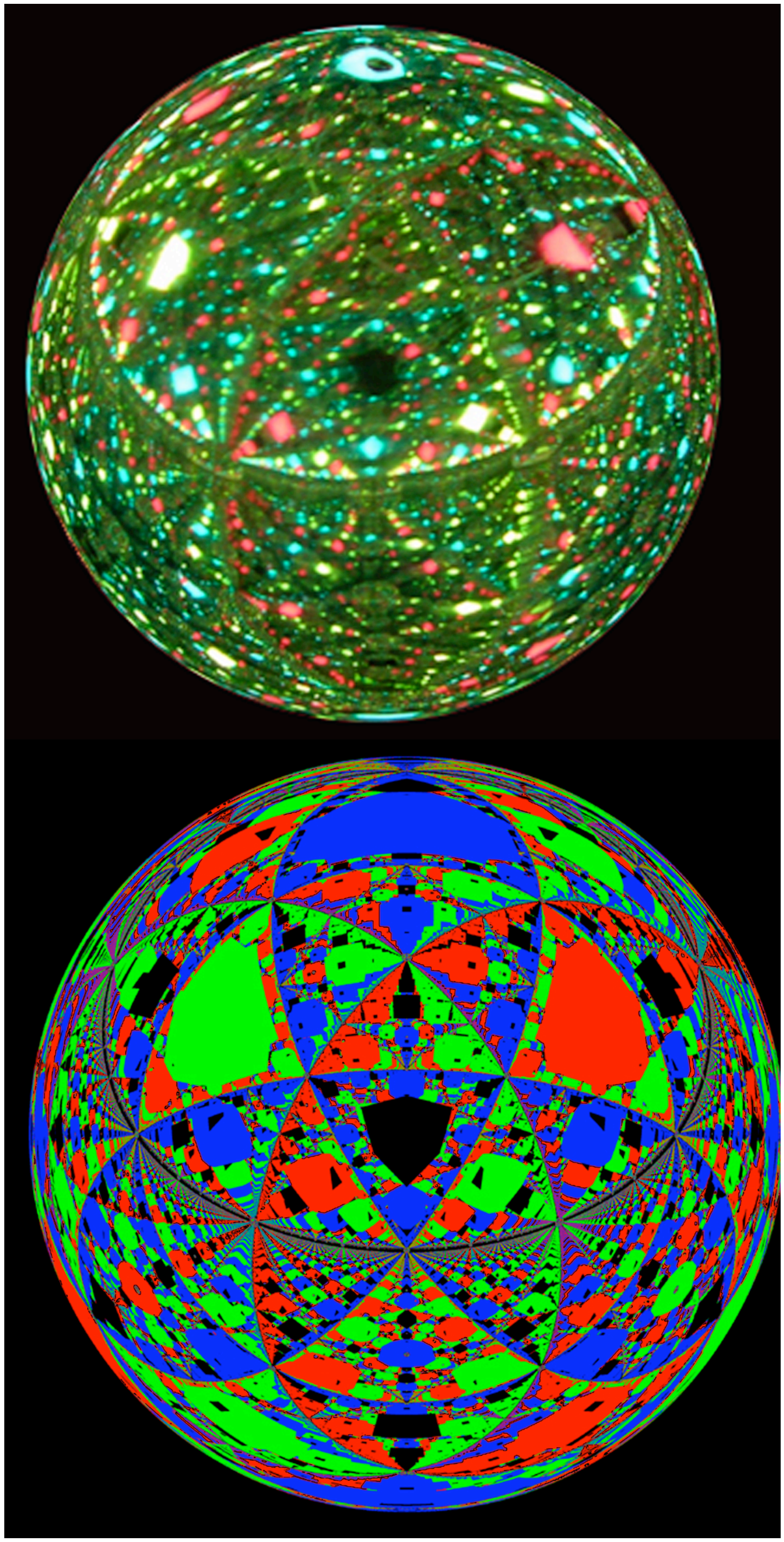

Figure 6: *(top) The image obtained from the opening of the Sinai cube and (bottom) the ray tracing model of that image.*

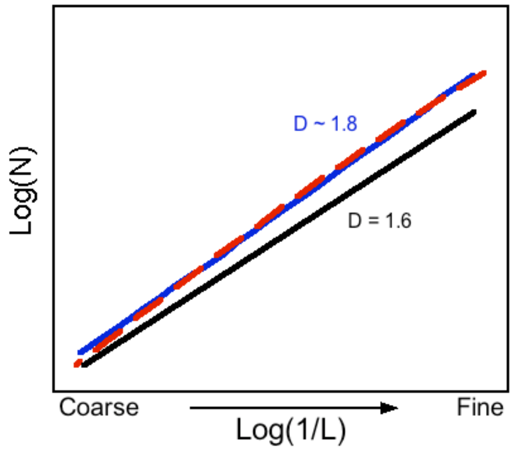

Figure 7: *Scaling plots of the images in figure 6 (upper dashed and solid lines respectively). The lower solid line depicts represents the scaling plot of the stacked sphere configuration.*

It is our hope that the inherent aesthetic quality of the images created by this simple demonstration as illustrated in Figure 8 can be used to motivate discussion of fractal geometry in high school and university math and physics curriculum. Study of fractal geometry offers high school and lower division college students a rare opportunity to investigate a field in which the first people to really think and elaborate about the subject are still alive.

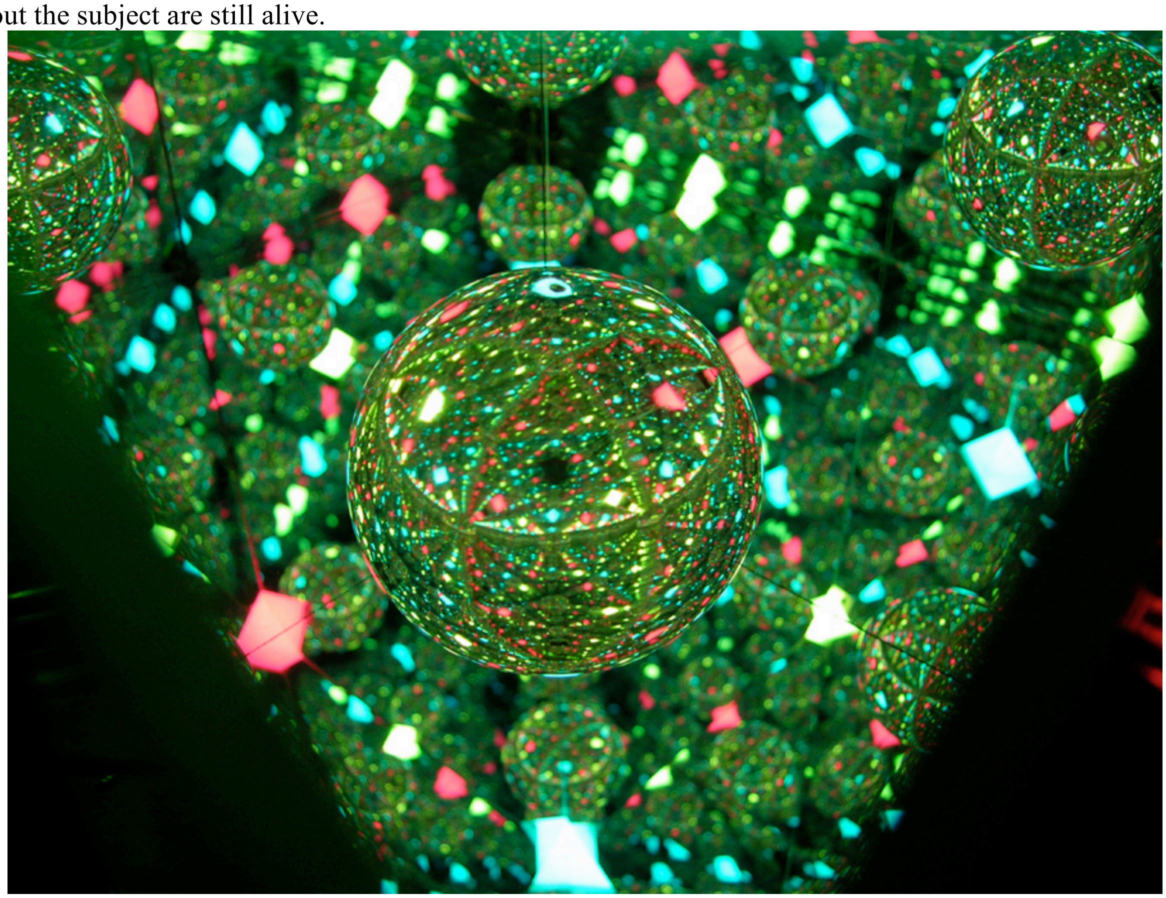

Figure 8: *The unaltered view from the empty top corner of the Sinai cube*

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References

[1] Benoit B. Mandelbrot, *The Fractal Geometry of Nature,* W.H. Freeman and Co., 1982.

[2] Peitgen, H.O. and Saupe, D., *The Science of Fractal Images,* Springer-Verlag, 1988.

[3] R. Ketzmerick, *Fractal conductance fluctuations in generic chaotic cavities,* Phys. Rev. B, Vol 54, pp.10841 1996

[4] R.P.Taylor, R. Newbury, A.S. Sachrajda, Y. Feng, P.T Coleridge, C. Dettmann, Ningjia Zhu, Hong Guo, A. Delage, P.J. Kelly and Z. Wasilewski, *Self –Similar Magnetoresitance of a Semiconductor Sinai Billiard,* Phys. Rev . Lett. Vol. 78, pp.1952, 1997.

[5] C.A. Marlow, R.P. Taylor, T.P> Martin, B.C. Scannell, H. Linke, M.S. Fairbanks, G.D. Hall, I. Shorubalko, L. Samuelson, T.M. Fromhold, C.V> Brown, B. Hackens, S> Faniel, C. Gustin, V. Bayot, X. Wallart, S. Bollaert, and A. Cappy, *Unified model of fractal conductance fluctuations for diffusive and ballistic semiconductor devices,* Phys. Rev. B, Vol. 73, 195318, 2006.

[6] D. Sweet, E. Ott, J.A. Yorke, *Topology in chaotic scattering*, Nature, Vol. 399, pp315, 1999

[7] Ray tracing was performed using a 3d modeling program called Cinema4D.