Applied Bayesian Modeling, ICSPR 2014, Prof. Ryan Bakker
Rose Maier, University of Oregon - Psychology
Assignment 1: Thursday, June 26, 2014

Due: Monday, June 30, 2014
On the first page of your submission, please indicate your *name*, *home institution*, and *field of study*.
Please turn in hard copies to the TAs during our course meeting on Monday, June 30.

1. Prove that the gamma distribution,

$$p(\theta|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}, \quad \theta,\alpha,\beta > 0 \tag{1}$$

is the conjugate prior distribution for θ in a Poisson probability mass function,

$$p(y_i|\theta) = \frac{e^{-\theta}\theta^{y_i}}{y_i!} \tag{2}$$

that is, calculate a form for the posterior distribution of θ and show that it is also gamma distributed.

First, convert the Poisson probability mass function to a likelihood function by multiplying the probability for each observation in the data together:

$$L(\theta|Y) = \prod_{i=1}^{n} \frac{e^{-\theta}\theta^{y_i}}{y_i!} \tag{3}$$

Move the terms with θ outside of the product operator.

$$L(\theta|Y) = e^{-n\theta}\theta^{\sum_{i=1}^{n} y_i} \prod_{i=1}^{n} \frac{1}{y_i!} \tag{4}$$

Multiply prior and likelihood together.

$$\Pi(\theta|Y) \propto \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}e^{-n\theta}\theta^{\sum_{i=1}^{n} y_i} \prod_{i=1}^{n} \frac{1}{y_i!} \tag{5}$$

Rearrange to put like terms next to each other.

$$\Pi(\theta|Y) \propto \frac{\beta^\alpha}{\Gamma(\alpha)\prod_{i=1}^{n} y_i!}e^{-\beta\theta}e^{-n\theta}\theta^{\alpha-1}\theta^{\sum_{i=1}^{n} y_i} \tag{6}$$

Combine like terms and drop the constant.

$$\Pi(\theta|Y) \propto e^{-(\beta+n)\theta}\theta^{\alpha+\sum_{i=1}^{n}(y_i)-1} \tag{7}$$

This looks like the kernel of a new gamma distribution, with $\alpha = \alpha + \sum_{i=1}^{n}(y_i)$ and $\beta = \beta + n$.

2. Now use the Gamma-Poisson conjugate specification to analyze data on the number of presidential appointments from 1960 to 2000. The data are in the file called `hw1.dta` on the Z-drive.

3. The posterior distribution for θ is gamma(δ_1, δ_2) according to some parameters δ_1 and δ_2 that you derived above which of course depend on your choice of the parameters for the gamma prior. You should model θ using two sets of priors. One which specifies a great deal of certainty regarding your best guess as to the value of θ and one that represents ignorance regarding this value.

   I choose $\delta_1 = 1, \delta_2 = 1$ for my uncertain prior, and $\delta_1 = 100, \delta_2 = 100$ for my very certain prior.

4. Generate a large number of values from this distribution in `R`, say $10,000$ or so, using the command:
   `posterior.sample <- rgamma(10000,d1,d2)`

5. Summarize the posteriors with quantities of interest such as means, medians, and variances. Also supply plots of the density of the posterior distributions.

```
setwd("/Users/TARDIS/Documents/ICPSR2014/Bayes/Homeworks/HW 1")
library(foreign)
df <- read.dta("hw1.dta")

n <- nrow(df)

# priors
d1_pri <- c(1,100)
d2_pri <- c(1,100)
mean_pri <- d1_pri / d2_pri
var_pri <- d1_pri / (d2_pri)^2

prior.sample1 <- rgamma(100000,shape=d1_pri[1],rate=d2_pri[1])
mean(prior.sample1)

## [1] 1.001

var(prior.sample1)

## [1] 0.998

prior.sample2 <- rgamma(100000,shape=d1_pri[2],rate=d2_pri[2])
mean(prior.sample2)

## [1] 1

var(prior.sample2)
```

```
## [1] 0.01002

# posteriors
d1_pos <- d1_pri + sum(df$appoints)
d2_pos <- d2_pri + n
mean_pos <- d1_pos / d2_pos
var_pos <- d1_pos / (d2_pos)^2

posterior.sample1 <- rgamma(100000,shape=d1_pos[1],rate=d2_pos[1])
mean(posterior.sample1)

## [1] 1.635

median(posterior.sample1)

## [1] 1.606

var(posterior.sample1)

## [1] 0.1485

posterior.sample2 <- rgamma(100000,shape=d1_pos[2],rate=d2_pos[2])
mean(posterior.sample2)

## [1] 1.064

median(posterior.sample2)

## [1] 1.061

var(posterior.sample2)

## [1] 0.00971

## plotting
plotdf1 <- data.frame(posterior.sample=posterior.sample1,
        prior.sample=prior.sample1)
plotdf1$prior <- as.factor(paste("prior d1=",d1_pri[1],
        " d2=",d2_pri[1], sep=""))

plotdf2 <- data.frame(posterior.sample=posterior.sample2,
        prior.sample=prior.sample2)
plotdf2$prior <- as.factor(paste("prior d1=",d1_pri[2],
        " d2=",d2_pri[2], sep=""))

plotdf <- rbind(plotdf1, plotdf2)
plotdf <- melt(plotdf, id.vars="prior")
```

```r
library(plyr)
means <- ddply(plotdf, .(prior, variable),
        summarize, means=mean(value))
plotdf <- merge(plotdf,means, by=c("prior", "variable"))

library(ggplot2)
ggplot(plotdf, aes(x=value)) +
  geom_histogram(position="identity",
        binwidth = .05, alpha=.5, aes(fill=variable)) +
  xlim(0,4) +
  geom_vline(data=plotdf,
        aes(xintercept=means, color=variable), lty=2) +
  facet_wrap(~prior)
```