

# The mean values of cubic $L$ -functions

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# The Riemann Zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} \quad \operatorname{Re}(s) > 1$$

- It can be analytically continued to  $\mathbb{C}$  with a pole at  $s = 1$ .
- The functional equation.

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

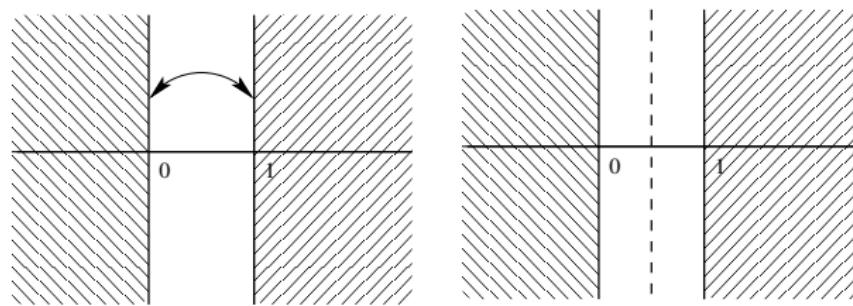
Then

$$\xi(s) = \xi(1-s).$$

# The Riemann Hypothesis

Trivial zeros:  $-2, -4, -6, \dots$

Nontrivial zeros:  $s = \sigma + i\tau$ ,  $0 < \sigma < 1$ .



The Riemann Hypothesis (RH) : nontrivial zeros of  $\zeta(s)$  at  $\operatorname{Re}(s) = \frac{1}{2}$ .

# The Lindelöf Hypothesis

The Lindelöf Hypothesis (LH) is a consequence of RH :

$$\zeta\left(\frac{1}{2} + it\right) = O(t^\varepsilon)$$

This is equivalent to

$$\frac{1}{X} \int_0^X \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} dt = O(X^\varepsilon) \quad \forall k \in \mathbb{N}$$

Understanding higher moments of  $\zeta(s)$  gives us progressively better bounds for  $\zeta(1/2 + it)$ .

# Moments of the Riemann zeta function - Some history

Let

$$I_k(X) = \int_0^X \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} dt.$$

- $k = 1$  Hardy & Littlewood (1918)

$$I_1(X) \sim X \log X$$

- $k = 2$  Ingham (1926)

$$I_2(X) \sim \frac{1}{2\pi^2} X (\log X)^4$$

- $k \geq 3$  remain **unsolved!** But there is a good conjecture...

## Higher moments of $\zeta(s)$ - conjectures

Keating and Snaith (2000), using Random Matrix Theory, conjectured

$$I_k(X) \sim \frac{a_k g_k}{(k^2)!} X (\log X)^{k^2},$$

where

$a_k$  is the leading coefficient of  $\sum_{n \leq X} \frac{d_k^2(n)}{n}$  (an Euler product)

$$d_k(n) = |\{n_1 \cdots n_k = n\}|,$$

and

$$g_k = (k^2)! \prod_{j=0}^{k-1} \frac{j!}{(k+j)!}.$$

The conjecture has been refined by several people, giving lower order terms. It was also recovered by Diaconu, Goldfeld, and Hoffstein (2003) with multiple Dirichlet series.



# Dirichlet characters

A **Dirichlet character** is

$$\chi : (\mathbb{Z}/d\mathbb{Z})^* \rightarrow \mathbb{C}^*$$

extended to  $\mathbb{Z}$  by periodicity and with the condition that

$$\chi(a) = 0 \text{ when } (a, d) \neq 1.$$

A Dirichlet character modulo  $d$  can be viewed as a character modulo  $D$  for any  $d \mid D$ .

$\chi$  is a **primitive character of conductor  $d$**  if  $d$  is the smallest modulo such that  $\chi$  is a character modulo  $d$ .

# Dirichlet $L$ -functions

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \quad \operatorname{Re}(s) > 1,$$

where  $\chi$  is a primitive Dirichlet character modulo  $d$ .

It can be analytically continued to  $\mathbb{C}$  and has a functional equation  
 $s \longleftrightarrow 1 - s$ .

## The moments

$$\sum_{\substack{\chi \bmod d \\ d \leq X}}^* L\left(\frac{1}{2}, \chi\right)^k$$

are associated with nonvanishing results.

There is a general philosophy that non-vanishing of  $L$ -functions at the critical point should be explained by arithmetic reasons.

# Moments of $\zeta(s)$ and $L(s, \chi)$

What is the relation between

$$\int_0^X \left| \zeta\left(\frac{1}{2} + it\right) \right|^k dt \text{ and } \sum_{\substack{\chi \bmod d \\ d \leq X}}^* L\left(\frac{1}{2}, \chi\right)^k ?$$

$$L\left(\frac{1}{2}, \chi\right) \leftrightarrow \sum \frac{\chi(n)}{n^{\frac{1}{2}}}$$
$$\zeta\left(\frac{1}{2} + it\right) \leftrightarrow \sum \frac{n^{-it}}{n^{\frac{1}{2}}}$$

(**Warning:** the right-hand sides do not make any sense!)

Notice:  $\varphi_t(n) = n^{-it}$  is “like” a character. Integrating is like summing over a continuous index.

# Moments of primitive quadratic Dirichlet $L$ -functions

As before, using random matrix theory, Keating and Snaith (2000) conjectured that

$$\sum_{\substack{\chi \text{ mod } d \\ \chi^2=1, d \leq X}}^* L\left(\frac{1}{2}, \chi\right)^k \sim a_k g_k X (\log X)^{\frac{k(k+1)}{2}}.$$

The  $*$  indicates primitive characters.

- $k = 1$  Jutila (1981)
- $k = 2$  Jutila (1981), Soundararajan (secondary main term, 2000)
- $k = 3$  Soundararajan (2000), Diaconu, Goldfeld, Hoffstein
- $k = 4$  Soundararajan and Young (under GRH, 2010), Shen

## Non-vanishing results

By Cauchy–Schwartz,

$$\begin{aligned} \sum^*_{\substack{\chi \text{ mod } d \\ \chi^2=1, d \leq X}} L\left(\frac{1}{2}, \chi\right) &\ll \\ \left( \sum^*_{\substack{\chi \text{ mod } d \\ \chi^2=1, d \leq X}} L\left(\frac{1}{2}, \chi\right)^2 \right)^{1/2} &(\# \{ \chi_d \text{ primitive, } d \leq X, L\left(\frac{1}{2}, \chi\right) \neq 0 \})^{1/2} \\ \# \{ \chi_d \text{ primitive, } d \leq X, L\left(\frac{1}{2}, \chi\right) \neq 0 \} &\gg \frac{\left( \sum^*_{\substack{\chi \text{ mod } d \\ \chi^2=1, d \leq X}} L\left(\frac{1}{2}, \chi\right) \right)^2}{\sum^*_{\substack{\chi \text{ mod } d \\ \chi^2=1, d \leq X}} L\left(\frac{1}{2}, \chi\right)^2} \\ &\gg \frac{X}{\log X} \sim X^{1-\varepsilon}. \end{aligned}$$

Soundararajan (2000) developed a finer method to prove that at least  $7/8$  of the  $L\left(\frac{1}{2}, \chi\right)$  do not vanish.

# Moments with cubic characters - $\mathbb{Q}$ (non-Kummer case)

Let  $p \equiv 1 \pmod{3}$ .

$\chi_p(a)$  is defined using  $a^{\frac{p-1}{3}} \pmod{p}$ .

We get two characters,  $\chi_p$  and  $\overline{\chi_p}$ .

$$\#\{\chi \pmod{d} : \chi^3 = 1, d \leq X\} \sim CX$$

Baier & Young (2010) proved,

$$\sum_{\substack{\chi \pmod{d \\ \chi^3=1, d \leq X}}}^* L\left(\frac{1}{2}, \chi\right) \sim cX + O\left(X^{\frac{37}{38}+\varepsilon}\right),$$

This implies a nonvanishing result of  $\gg X^{\frac{6}{7}-\varepsilon}$ .



# Moments with cubic characters - $\mathbb{Q}(\xi_3)$ (Kummer case)

Over  $\mathbb{Q}(\xi_3)$ , for each prime  $\mathfrak{p}$ , we have  $\chi_{\mathfrak{p}}$  and  $\overline{\chi_{\mathfrak{p}}}$ , where

$\chi_{\mathfrak{p}}(a)$  is defined using  $a^{\frac{N(\mathfrak{p})-1}{3}} \pmod{\mathfrak{p}}$ .

$$\#\{\chi \pmod{d} : \chi^3 = 1, N(d) \leq X\} \sim CX \log X$$

Luo (2004)

$$\sum_{\substack{\chi \pmod{d} \\ d \in \mathbb{Z}[\xi_3], d \equiv 1 \pmod{9} \\ d \text{ square-free}, N(d) \leq X}}^* L\left(\frac{1}{2}, \chi\right) \sim cX + O\left(X^{\frac{21}{22} + \varepsilon}\right),$$

This considers only a thin subset of the cubic characters, as it does not consider  $\overline{\chi_{\mathfrak{p}}}$ .

This implies a nonvanishing result of  $\gg X^{1-\varepsilon}$ .

# Function fields

Let  $q$  power of a prime,  $\mathbb{F}_q$  finite field with  $q$  elements.

Number Fields

$$\mathbb{Q}$$

$\leftrightarrow$

Function Fields

$$\mathbb{F}_q(X)$$

$$\mathbb{Z}$$

$\leftrightarrow$

$$\mathbb{F}_q[X]$$

$p$  positive prime  $\leftrightarrow P(X)$  monic irreducible polynomial

$|n| = |\mathbb{Z}/n\mathbb{Z}| = n \in \mathbb{N} \leftrightarrow |F(X)| = |\mathbb{F}_q[X]/(F(X))| = q^{\deg F}$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$\leftrightarrow$

$$\zeta_q(s) = \sum_{\substack{F \in \mathbb{F}_q[X] \\ F \text{ monic}}} \frac{1}{|F|^s}$$

Riemann Hypothesis 

$\leftrightarrow$

Riemann Hypothesis 

# The Riemann zeta function over finite fields

$$\zeta_q(s) = \sum_{\substack{f \in \mathbb{F}_q[X] \\ f \text{ monic}}} \frac{1}{|f|^s} = \prod_{\substack{\ell \text{ irreducible} \\ \text{monic}}} \left(1 - \frac{1}{|\ell|^s}\right)^{-1}$$

$f(X) = a_0 + a_1 X + \cdots + a_{d-1} X^{d-1} + X^d \Rightarrow q^d$  possibilities.

$$\begin{aligned} \zeta_q(s) &= \sum_{\substack{f \in \mathbb{F}_q[X] \\ f \text{ monic}}} \frac{1}{|f|^s} = \sum_{\substack{f \in \mathbb{F}_q[X] \\ f \text{ monic}}} \frac{1}{q^{s \deg f}} \\ &= \sum_{d=0}^{\infty} \frac{q^d}{q^{sd}} = \frac{1}{1 - q^{1-s}} \end{aligned}$$

$\zeta_q(s)$  has no zeros, the Riemann hypothesis is true!



# Dirichlet $L$ -functions over function fields

A Dirichlet character is a

$$\chi : (\mathbb{F}_q[X]/(D(X)))^* \rightarrow \mathbb{C}^*$$

extended to  $\mathbb{F}_q[X]$  by periodicity and with the condition that

$$\chi(A) = 0 \text{ when } (A, D) \neq 1.$$

The Dirichlet  $L$ -function is

$$L(s, \chi) = \sum_{f \text{ monic}} \frac{\chi(f)}{|f|^s} = \sum_{n=0}^{\infty} \frac{1}{q^{ns}} \sum_{\deg f = n} \chi(f).$$

Since  $\sum_{\deg f = n} \chi(f) = 0$  for  $n \geq \deg D$ , we get a finite sum with finitely many zeros:

$$L(s, \chi) = \sum_{\deg f < \deg D} \frac{\chi(f)}{|f|^s}.$$

# Quadratic Dirichlet characters over function fields

For a monic irreducible  $P$ , and  $f \in \mathbb{F}_q[X]$ , define

$$\left(\frac{f}{P}\right) = \begin{cases} 1, & P \nmid f, f \equiv \square \pmod{P} \\ -1, & P \nmid f, f \not\equiv \square \pmod{P} \\ 0, & P \mid f. \end{cases}$$

If  $Q = P_1^{e_1} \cdots P_k^{e_k}$ , the Jacobi symbol is

$$\left(\frac{f}{Q}\right) = \prod_{i=1}^k \left(\frac{f}{P_i}\right)^{e_i}.$$

Quadratic Reciprocity: If  $A, B \in \mathbb{F}_q[X]$  are non-zero, relatively prime monic,

$$\left(\frac{A}{B}\right) \left(\frac{B}{A}\right) (-1)^{\left(\frac{q-1}{2}\right) \deg A \deg B}.$$

# Quadratic Dirichlet $L$ -function over function fields

Assume  $q \equiv 1 \pmod{4}$ . Write

$$\left(\frac{D}{f}\right) = \chi_D(f)$$

$$L(s, \chi_D) = \sum_{f \text{ monic}} \frac{\chi_D(f)}{|f|^s} = \prod_{\substack{P \text{ irreducible} \\ \text{monic} \\ P \nmid D}} \left(1 - \frac{\chi_D(P)}{|\ell|^s}\right)^{-1}$$

Make the change  $u = q^{-s}$ .

$$\mathcal{L}(u, \chi_D) = \sum_{f \text{ monic}} \chi_D(f) u^{\deg f} = \sum_{n=0}^{\infty} u^n \sum_{\substack{f \text{ monic} \\ \deg f = n}} \chi_D(f).$$

# Quadratic Dirichlet $L$ -function over function fields

Consider the hyperelliptic curve

$$C_D : Y^2 = D(X),$$

where  $D(x)$  is monic and  $\square$ -free.

$$Z_{C_D}(u) = \exp \left( \sum_{k=1}^{\infty} |C(\mathbb{F}_{q^k})| \frac{u^k}{k} \right), \quad |u| < 1/q,$$

If  $\deg D = 2g + 1$ ,

$$Z_{C_D}(u) = \frac{\mathcal{L}(u, \chi_D)}{(1-u)(1-qu)}.$$

The Weil conjectures imply

- $\mathcal{L}(u, \chi_D)$  is a polynomial of degree  $g$ .
- Functional equation relating  $\mathcal{L}(u, \chi)$  and  $\mathcal{L}\left(\frac{1}{qu}, \overline{\chi}\right)$ .
- The Riemann Hypothesis holds, the zeros are at  $|u| = \frac{1}{\sqrt{q}}$ .

# Moments of quadratic Dirichlet $L$ -functions over function fields

Andrade and Keating (2014) conjectured

$$\sum_{D \in \mathcal{H}_{2g+1}} L\left(\frac{1}{2}, \chi_D\right)^k = q^{2g+1} P_k(2g+1) + o(q^{2g+1}),$$

where  $P_k$  is a polynomial of degree  $\frac{k(k+1)}{2}$ .

They proved this for  $k = 1$ .

Florea (2017, several papers) gave the second order term for  $k = 1$  and proved cases for  $k = 2, 3, 4$ .

Bui and Florea (2016) nonvanishing for  $\geq 94\%$ .

Li (2018) vanishing  $\gg X^{\frac{1}{3}-\varepsilon}$ .

# Cubic characters over function fields (Kummer case)

Let  $q \equiv 1 \pmod{3}$  odd and fix

$$\Omega : \text{roots of unity in } \mathbb{C}^* \rightarrow \text{roots of 1 in } \mathbb{F}_q^*.$$

For a monic irreducible  $P$ , and  $f \in \mathbb{F}_q[T]$  such that  $P \nmid f$ , define

$$f^{\frac{q^{\deg(P)} - 1}{3}} \equiv \Omega(\alpha) \pmod{P} \quad \chi_P(f) := \alpha$$

and for  $P \mid f$ ,

$$\chi_P(f) = 0.$$

If  $Q = P_1^{e_1} \cdots P_k^{e_k}$ , the Jacobi symbol is

$$\chi_Q(f) = \prod_{i=1}^k \chi_{P_i}(f)^{e_i}.$$

# The Weil Zeta function (Kummer case)

$q \equiv 1 \pmod{6}$ . Consider

$$C_{F_1, F_2} : Y^3 = F_1(T)F_2(T)^2,$$

where  $F_1, F_2$  are  $\square$ -free and relatively prime.

Let  $\deg F_i = d_i$  with  $d_1 + 2d_2 \equiv 1 \pmod{3}$

$$\mathcal{L}\left(u, \chi_{F_1 F_2^2}\right) = \sum_{\deg f \leq d_1 + d_2 - 1} \chi_{F_1 F_2^2}(f) u^{\deg f}.$$

$$Z_{C_{F_1, F_2}}(u) = \frac{\mathcal{L}(u, \chi_{F_1 F_2^2}) \mathcal{L}(u, \overline{\chi_{F_1 F_2^2}})}{(1-u)(1-qu)}.$$

$$d_1 + d_2 = g + 1$$

# The mean value of cubic Dirichlet $L$ -functions over function fields (Kummer case)

Theorem (David, Florea, L. (2019+))

Let  $q$  be an odd prime power such that  $q \equiv 1 \pmod{3}$ . Let  $\chi_3$  be a fixed cubic character on  $\mathbb{F}_q^*$

$$\sum_{\substack{\chi \text{ primitive cubic} \\ \text{genus}(\chi)=g \\ \chi|_{\mathbb{F}_q^*}=\chi_3}} L\left(\frac{1}{2}, \chi\right) = C_1 g q^{g+1} + C_2 q^{g+1} + O\left(q^{g \frac{1+\sqrt{7}}{4} + \varepsilon g}\right),$$

where  $C_1$  and  $C_2$  are certain constants and  $g = \deg(\text{Cond}(\chi)) - 1$ .

$$\#\{\chi : \text{genus}(\chi) = g\} = B_1 g q^{g+1} + B_2 q^{g+1} + O\left(q^{(\frac{1}{2}+\varepsilon)g}\right)$$



## Some features

- The size of the family,  $gq^g$ , corresponds to  $X \log X$ . Baier & Young had size  $X$ .
- The shape of the main term is strange.
- The sieve is complicated.
- Difficulty estimating the error when one  $d_i$  is small.
- This implies a nonvanishing result of  $\gg q^{g(1-\varepsilon)}$ .

# Cubic characters over function fields (non-Kummer case)

Let  $q \equiv 2 \pmod{3}$  odd and fix

$$\Omega : \text{roots of unity in } \mathbb{C}^* \rightarrow \text{roots of 1 in } \mathbb{F}_{q^2}^*.$$

For a monic irreducible  $P$  of **even degree** and  $f \in \mathbb{F}_q[T]$  such that  $P \nmid f$ , define

$$f^{\frac{q^{\deg(P)} - 1}{3}} \equiv \Omega(\alpha) \pmod{P} \quad \chi_P(f) := \alpha$$

Write  $P = \pi\tilde{\pi}$  in  $\mathbb{F}_{q^2}[T]$  and get  $\chi_\pi, \chi_{\tilde{\pi}}$  characters in  $\mathbb{F}_{q^2}[T]$ . Then

$$\chi_P = \chi_\pi|_{\mathbb{F}_q[T]}.$$

Extend as before...

# The Weil Zeta function (non-Kummer case)

$q \equiv 2 \pmod{3}$ . Consider

$$C_{F_1, F_2} : Y^3 = F_1(T)F_2(T)^2,$$

where  $F_1, F_2$  are  $\square$ -free and relatively prime.  $P \mid F_i \Rightarrow \deg(P)$  even.

Let  $\deg F_i = d_i$  with  $d_1 + 2d_2 \equiv 0 \pmod{3}$

$$Z_{C_{F_1, F_2}}(u) = \frac{\mathcal{L}^*(u, \chi_{F_1 F_2^2}) \mathcal{L}^*(u, \overline{\chi_{F_1 F_2^2}})}{(1-u)(1-qu)}.$$

$$\mathcal{L}^*(u, \chi_{F_1 F_2^2}) = \frac{\mathcal{L}(u, \chi_{F_1 F_2^2})}{1-u}$$

$$d_1 + d_2 = g + 2$$

# The mean value of cubic Dirichlet $L$ -functions over function fields (non-Kummer case)

Theorem (David, Florea, L. (2019+))

Let  $q$  be an odd prime power such that  $q \equiv 2 \pmod{3}$ . Then

$$\sum_{\substack{\chi \text{ primitive cubic} \\ \text{genus}(\chi)=g}} L\left(\frac{1}{2}, \chi\right) = Aq^{g+2} + O\left(q^{\frac{7g}{8}+\varepsilon g}\right),$$

where  $A$  is certain given constant and  $g = \deg(\text{Cond}(\chi)) - 2$ .

$$\#\{\chi : \text{genus}(\chi) = g\} = Dq^{g+2} + O\left(q^{\left(\frac{1}{2}+\varepsilon\right)g}\right)$$

if  $2 \mid g$  and zero otherwise.

Our results imply nonvanishing of  $\gg q^{g(1-\varepsilon)}$ .

## Ideas in the proof - Approximate functional equation

$$\mathcal{L}^*(u, \chi) = \omega(\chi)(\sqrt{q}u)^g \mathcal{L}^* \left( \frac{1}{qu}, \bar{\chi} \right),$$

- $\chi$  odd “ $q \equiv 1 \pmod{3}$ ”

$$\mathcal{L} \left( \frac{1}{\sqrt{q}}, \chi \right) = \underbrace{\sum_{f \in \mathcal{M}_{q, \leq A}} \frac{\chi(f)}{q^{\deg(f)/2}}}_{\sum_{\chi} \rightsquigarrow S_{\text{principal}}} + \omega(\chi) \underbrace{\sum_{f \in \mathcal{M}_{q, \leq g-A-1}} \frac{\overline{\chi(f)}}{q^{\deg(f)/2}}}_{\sum_{\chi} \rightsquigarrow S_{\text{dual}}}.$$

- $\chi$  even “ $q \equiv 2 \pmod{3}$ ”

$$\begin{aligned} \mathcal{L} \left( \frac{1}{\sqrt{q}}, \chi \right) &= \sum_{f \in \mathcal{M}_{q, \leq A}} \frac{\chi(f)}{q^{\deg(f)/2}} + \omega(\chi) \sum_{f \in \mathcal{M}_{q, \leq g-A-1}} \frac{\overline{\chi(f)}}{q^{\deg(f)/2}} \\ &\quad + \frac{1}{1 - \sqrt{q}} \sum_{f \in \mathcal{M}_{q, A+1}} \frac{\chi(f)}{q^{\deg(f)/2}} + \frac{\omega(\chi)}{1 - \sqrt{q}} \sum_{f \in \mathcal{M}_{q, g-A}} \frac{\overline{\chi(f)}}{q^{\deg(f)/2}}. \end{aligned}$$

## Ideas in the proof - The principal term

If  $f = \square$ ,

$$S_{\text{principal}} = \sum_{f=\square} + \sum_{f \neq \square}, \quad \chi_F(f) = \begin{cases} 1 & (F, f) = 1, \\ 0 & (F, f) \neq 1. \end{cases}$$

$$\sum_{f=\square} = \sum_{\deg(f) \leq A/3} \frac{a(F)}{|f|^{3/2}},$$

$$a(F) = \#\{\chi : \chi^3 = 1, \text{ primitive, } \text{Cond}(\chi) = F\}$$

with  $(F, f) = 1$ .

## Ideas in the proof - The main term

$$a(F) = \#\{\chi : \chi^3 = 1, \text{ primitive, } \text{Cond}(\chi) = F\}$$

Let

$$\mathcal{G}(u) = \sum_F a(F) u^{\deg(F)} = \prod_{\substack{2|\deg(P) \\ P \nmid f}} (1 + 2u^{\deg(P)}) \text{ non-Kummer}$$

By Perron's formula,

$$\sum_{\deg(F)=d} a(F) = \frac{1}{2\pi i} \oint_{|u|=q^{-2}} \frac{\mathcal{G}(u)}{u^d} \frac{du}{u}$$

$$\sum_{f=\square} = M_g q^g + N_g q^{g-\frac{A}{6}} + O\left(q^{g-\frac{A}{2}}\right)$$

$\sum_{f \neq \square} \ll q^{\frac{A+g}{2} + \varepsilon g}$  uses the Lindelöf bound.

## Ideas in the proof - The dual sum

$$\mathcal{L}^*(u, \chi) = \omega(\chi)(\sqrt{q}u)^g \mathcal{L}^* \left( \frac{1}{qu}, \bar{\chi} \right),$$

$$\mathcal{L} \left( \frac{1}{\sqrt{q}}, \chi \right) = \underbrace{\sum_{f \in \mathcal{M}_{q, \leq A}} \frac{\chi(f)}{q^{\deg(f)/2}}}_{\sum_{\chi \rightsquigarrow S_{\text{principal}}}} + \underbrace{\omega(\chi) \sum_{f \in \mathcal{M}_{q, \leq g-A-1}} \frac{\overline{\chi(f)}}{q^{\deg(f)/2}}}_{\sum_{\chi \rightsquigarrow S_{\text{dual}}} \overline{\chi(f)}}$$

$$\omega(\chi) = q^{-g/2-1} G(\chi)$$

$$G(\chi_F) = \sum_{a \bmod F} \chi_F(a) e \left( \frac{a}{F} \right)$$

Cubic Gauss sum!

## Ideas in the proof - Gauss sums

If  $a \in \mathbb{F}_q((\frac{1}{t}))$ , define,

$$e(\alpha) = e^{\frac{2\pi i \text{tr}(a_1)}{p}}, \text{ if } \alpha = \cdots + \frac{a_1}{t} + \cdots$$

Gauss sum

$$G(V, F) = \sum_{a \bmod F} \chi_f(a) e\left(\frac{aV}{F}\right)$$

“almost” multiplicative.

- If  $(F_1, F_2) = 1$ ,

$$G(V, F_1 F_2) = G(V, F_1) G(V, F_2) \chi_{F_1}^2(F_2)$$



$$P \nmid V \Rightarrow |G(V, P)| = \sqrt{|P|}.$$

## Ideas in the proof - The sum of the Gauss sums

Hoffstein (1992) and Patterson (2007) studied

$$\Psi_f(u) = \sum_F G(f, F) u^{\deg F}$$

(from the context of metaplectic Eisenstein series).

Use Perron formula  $\Psi_f(u) \longleftrightarrow \Psi_f\left(\frac{1}{q^2 u}\right)$  ( $s \leftrightarrow 2 - s$ ), poles at  $u^3 = \frac{1}{q^4}$ .

$$\sum_{\substack{F \text{ monic} \\ \deg F = d \\ (F, f) = 1}} G(f, F) = \frac{1}{2\pi i} \oint_{|u|=q^{-2}} \frac{\Psi_f(u) \tilde{\Psi}_f(u)}{u^d} \frac{du}{u}$$

Extra poles at  $u^3 = \frac{1}{q^2}$



## Ideas in the proof - Combining with the dual sum

$$\sum_{f=\square} = M_g q^g + N_g q^{g-\frac{A}{6}} + O\left(q^{g-\frac{A}{2}}\right), \quad \sum_{f \neq \square} \ll q^{\frac{A+g}{2} + \varepsilon g}$$

- For non-Kummer:

$$S_{\text{dual}} = -N_g q^{g-\frac{A}{6}} + O_g\left(q^{g-\frac{A}{2}} + q^{\frac{3g}{2} - (2-\sigma)A} + q^{\frac{5g}{6}}\right)$$

Take  $\sigma = 7/6$ ,  $A = 3g/4$ , then  $O\left(q^{\frac{7g}{8} + \varepsilon g}\right)$ .

- For Kummer:

$$S_{\text{dual}} = -N_g q^{g-\frac{A}{6}} + O_g\left(q^{g-\frac{A}{6}} + q^{\left(\frac{23}{12} - \frac{\sigma}{2} + \varepsilon\right)g - \left(\frac{13}{12} - \frac{\sigma}{2}\right)A} + q^{\frac{3g}{2} - (2-\sigma)A} + q^{\frac{5g}{6}}\right)$$

Take  $\sigma = \frac{13-2\sqrt{7}}{6}$ ,  $A = \frac{\sqrt{7}-1}{2}g$ , then  $O\left(q^{g\frac{1+\sqrt{7}}{4} + \varepsilon g}\right)$ .

## The results

Theorem (David, Florea, L. (2019+))

Let  $q$  be an odd prime power such that  $q \equiv 1 \pmod{3}$ . Let  $\chi_3$  be a fixed cubic character on  $\mathbb{F}_q^*$

$$\sum_{\substack{\chi \text{ primitive cubic} \\ \text{genus}(\chi)=g \\ \chi|_{\mathbb{F}_q^*}=\chi_3}} L(1/2, \chi) = C_1 g q^{g+1} + C_2 q^{g+1} + O\left(q^{g \frac{1+\sqrt{7}}{4} + \varepsilon g}\right).$$

Theorem (David, Florea, L. (2019+))

Let  $q$  be an odd prime power such that  $q \equiv 2 \pmod{3}$ . Then

$$\sum_{\substack{\chi \text{ primitive cubic} \\ \text{genus}(\chi)=g}} L(1/2, \chi) = A q^{g+2} + O\left(q^{\frac{7g}{8} + \varepsilon g}\right).$$

# The conjectures

## Conjecture

Let  $q$  be an odd prime power such that  $q \equiv 1 \pmod{3}$ . Let  $\chi_3$  be a fixed cubic character on  $\mathbb{F}_q^*$

$$\sum_{\substack{\chi \text{ primitive cubic} \\ \text{genus}(\chi)=g \\ \chi|_{\mathbb{F}_q^*} = \chi_3}} L(1/2, \chi) = C_1 g q^{g+1} + C_2 q^{g+1} + D_1 g q^{\frac{5g}{6}} + D_2 q^{\frac{5g}{6}} + o\left(q^{\frac{5g}{6}}\right).$$

## Conjecture

Let  $q$  be an odd prime power such that  $q \equiv 2 \pmod{3}$ . Then

$$\sum_{\substack{\chi \text{ primitive cubic} \\ \text{genus}(\chi)=g}} L(1/2, \chi) = A q^{g+2} + B q^{\frac{5g}{6}} + o\left(q^{\frac{5g}{6}}\right).$$

# What about higher moments?

Conjecture

$$\frac{1}{\#\{\chi : \text{genus}(\chi) = g\}} \sum_{\substack{\chi \text{ primitive cubic} \\ \text{genus}(\chi)=g}} L(1/2, \chi)^k \sim C$$

## What about higher moments?

Let  $E$  be an elliptic curve. Let  $\mathcal{F}_E(D)$  be the set of primitive cubic characters over  $\mathbb{Q}$  with conductor  $\leq D$  coprime to  $3N_E$ .

$$L(s, E) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad L(s, E, \chi) = \sum_{n=1}^{\infty} \frac{a_n \chi(n)}{n^s}$$

Conjecture (David, L, Nam (2019+))

For  $k, \ell \in \mathbb{C}$ ,  $\operatorname{Re}(k), \operatorname{Re}(\ell) > -1$ ,

$$\frac{1}{\#\mathcal{F}_E(D)} \sum_{\chi \in \mathcal{F}_E(D)} L\left(\frac{1}{2}, E, \chi\right)^k L\left(\frac{1}{2}, E, \bar{\chi}\right)^{\ell} \sim \frac{g_{k,\ell} a_{k,\ell}}{\Gamma(k\ell + 1)} (\log D)^{k\ell}$$

The conjecture follows the recipe developed by Conrey, Farmer, Keating, Rubinstein, and Snaith (2005).



# Thanks for your attention!